M.Sc. (Mathematics) Part – II
Algebra – II
Paper- I
April - 2015

QP Code : 50362

Scheme A (External)]	(3 Hours)	**[Total Marks:100**
Scheme B (Internal)]	(2 Hours)	**[Total Marks: 40**

Instructions:

- **Scheme A** students should attempt **any five** questions.
- **Scheme B** students should attempt **any three** questions.
- **All** questions carry **equal marks.**
- **Mention** clearly the **Scheme** under which you are appearing.

1. (a) Let G be a finite group of order $n = prq$ where p is a prime and $(p, q) = 1$. Prove that, there exists a subgroup of G having order pj for every $j, 1 \leq j \leq r$.

   (b) Show that a group of order 105 cannot be simple.

2. (a) State the Jorden-Holder Theorem. Write all composition series of $\mathbb{Z}_{30}$. Verify the Jordan-Holder Theorem for $\mathbb{Z}_{30}$.

   (b) Show that a group of order 35 is solvable.

3. (a) Let F be a field and $f(x)$ be an irreducible polynomial over F. Prove that there exists an extension field of F in which $f(x)$ has a root.

   (b) Show that every finite field extension is an algebraic extension.

4. (a) Define cyclotomic polynomial $\phi(x)$ and show that it is an irreducible polynomial in $\mathbb{Q}[X]$.

   (b) Show that a regular n-gon is constructible iff $\phi(n)$ is a power of 2, where $\phi(n)$ is number of positive integers less than n and relatively prime to n.

5. (a) Let E be the splitting field of a polynomial $f(x)$ over a field F. Show that $|Aut(E/F) \leq [E : F]$ where $Aut(E/F)$ denotes the group of automorphisms of field E which fixes F element wise.

   (b) Prove or Disprove: There exists a field having 48 elements.

6. (a) Let $f(x)$ be a monic polynomial in $F[x]$.Prove that $f(x)$ has a multiple root in a splitting field of $f(x)$ iff its derivative $f'(x) = 0$. Hence show that if char $F = 0$, then $f(x)$ has no multiple roots and if char $F = p > 0$ then $f(x)$ has multiple roots iff $f(x)$ is in $F[x^p]$

   (b) Let F be a finite field with p^n elements. Show that F has a subfield F' with p^m elements iff m divides n.

7. (a) If N is a submodule of M with N and M/N finitely generated modules, show that M is also finitely generated.

   (b) R is a commutative ring with unity. M is an R-module. Show that the following are equivalent: (i) Descending chain condition holds in M (ii) Every quotient module is finitely generated.

8. (a) Show that a submodule of a finitely generated free module of rank n over a PID is free module of rank $\leq n$.

   (b) Define free module and torsion module. Give an example of a free module which is not torsion-free. Justify your answer.

M.Sc. (Mathematics) Part – II
Analysis – II
Paper- II
April - 2015

QP Code : 50408

Scheme A (External)]	(3 Hours)	**[Total Marks:100**
Scheme B (Internal)]	(2 Hours)	**[Total Marks: 40**

Instructions:

- **Scheme A** students should attempt **any five** questions.
- **Scheme B** students should attempt **any three** questions.
- **All** questions carry **equal marks.**
- **Mention** clearly the **Scheme** under which you are appearing.

1. (a) Define Lebesgue outer measure $m^*(X)$ of a subset X of $\mathbb{R}$. If X, Y are subsets of $\mathbb{R}$ with $X \subset Y$ then show that $m^*(X) \leq m^*(Y)$.

   (b) If A and B are two Lebesgue measurable subsets of $\mathbb{R}$ then prove that $A \cup B$ is also Lebesgue measurable.

2. (a) Let $E \subseteq \mathbb{R}$. Prove that E is Lebesgue measurable iff for any $\epsilon > 0$, we can find an open subset G of $\mathbb{R}$ such that $E \subseteq G$ and $m^*(G \backslash E) < \epsilon$.

   (b) Show that there exists a subset of $\mathbb{R}$ which is not Lebesgue measurable. Justify your answer.

3. (a) State and prove Fatou's lemma.

   (b) If f, g are real valued measurable functions on $\mathbb{R}$, show that the set $\{x \in \mathbb{R} : f(x)g(x) \neq 0\}$ is measurable.

4. (a) Define simple function. If f is a bounded measurable function on a measurable set E of finite measure, show that $\inf\limits_{f \leq \psi} \int\limits_E \psi(x)\, dx = \sup\limits_{f \geq \phi} \int\limits_E \phi(x)\, dx$ for all simple functions ψ and ϕ.

   (b) Give an example to show that the inequality in Fatou's lemma can be strict. Justify your answer.

5. (a) State and prove Lebesgue dominated convergence theorem.

   (b) Show that $\lim\limits_{n \to \infty} \int\limits_0^1 x^n (\sin x)^{200} dx = 0$.

6. (a) Let $E \subset \mathbb{R}$ with Lebesgue measure zero. Prove that any real valued function f on E is measurable and Lebesgue integrable with $\int\limits_E f = 0$.

   (b) If $p \geq 1$ and $f, g \in L^p(E)$ then show that $||f + g||_p \leq ||f||_p + ||g||_p$

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7. (a) Show that the set $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots\right\}$ forms an orthonormal system on $[-\pi, \pi]$.

   (b) If a Hilbert space H has a countable orthonormal basis, then show it is separable.

8. (a) State and prove Riemann-Lebesgue Lemma.

   (b) Find the solution to the Heat equation: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ satisfying $u(x, 0) = x$, $0 \leq x \leq \pi$ and $u(0, t) = 0 = u(\pi, t),\ t \geq 0$, explaining the method of proof.

M.Sc. (Mathematics) Part – II
Differential Geometry
Paper - III
April - 2015

QP Code : **50522**

(OLD COURSE)
(3 Hours) **[Total Marks : 100**

N.B. (1) Attempt any **five** questions.
(2) **All** questions carry **equal** marks.

1. (a) Describe the notion of orientation of on Euclidean space, How many orientations are there in IR^n ?
When an isometry T of IR^n is said to be orientation presening ?
Describe the classification of isometries of IR^n.
(b) Prove that the set of all orientation preserving isometries of IR^n form a group.

2. (a) Let U be an open subset of IR^n and let f be any vector field on U.
(i) When f is locally Lipschitz ? State and prove a sufficient condition for f to be locally Lipschitz.
(ii) Describe without proof Picard's scheme for the approximate solutions of the I.V.P. :

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0$$

$t_0 \in IR, \quad x_0 \in U$ being arbitrary.
(b) Obtain approximate solutions upto t^4 of the I.V.P. :

$$\left\{\begin{matrix} \frac{dx}{dt} = 2x + t, & x(0) = 2 \\ \frac{dy}{dt} = tx + y & y(0) = 3 \end{matrix}\right\}$$

3. (a) Let $C : (a,b) \to IR^3$ be a smooth curve with the property that $\dot{c}(t) \neq 0, \quad \ddot{c}(t) \neq 0$ for all $t \in (a,b)$.
Derive the formulae :

$$k(t) = \frac{\| \dot{c}(t) \times \ddot{c}(t) \|}{\| \dot{c}(t) \|^{3/2}}$$

$$\tau(t) = \frac{\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t))}{\| \dot{c}(t) \times \ddot{c}(t) \|^2}$$

for the curvature k(t) and the torsion $\tau(t)$ at a point c(t) of the curve c.

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(b) Calculate $k(t)$, $\tau(t)$ for the curves :

(i) $C(t) = (3t - t^3 \ 3t^2 \ 3t + t^3) \quad t \in IR.$

(ii) $C(r) = \left(\frac{3}{10}\cos 2r, \frac{2}{5}\cos 2r, \frac{1}{2}\sin 2r - 2\right) \quad r \in IR.$

4. (a) Let $c : (a,b) \to IR^3$ be a smooth curve and $t_0 \in (a,b)$ such that $\dot{c}(t_0) \neq 0$, $\ddot{c}(t_0) \neq 0$. Prove that when t_1, t_2, t_3 all in (a,b) are sufficiently near to t_0 the points $c(t_1)$, $c(t_2)$ $c(t_3)$ are not collinear and therefore they determine a cirlce $S(t_1, t_2, t_3)$. Prove that the circle $S(t_1, t_2, t_3)$ takes a limiting position when $t_1 \to t_0$, $t_2 \to t_0$, $t_3 \to t_0$. What is the radius of this circle ?

(b) Let $c : (a,b) \to IR^3$ be a smooth curve all the tangent lines of which pass through a fixed point. Prove that the curve is a part of a straight line.

5. (a) Let ω be an exterior three-form on an open subset U of IR^3 and let K be a compact subset of U. Define the integral $\int_K \omega$ and prove that the integration is a well-defined linear map on the vector space Ω^3 (U) of exterior three forms.

(b) Explain the generalized stokes theorem for integration of exterior forms.

6. Let M be a smooth surface and let p be a point of it.

(a) Define (i), the first fundamental form I(p) of M at p, (ii) the Weingarten map $Lp : Tp(M) \to Tp(M)$ at p and prove that Lp is self-adjoint.

(b) Let $f. U \to IR$ be a smooth function defined on an open subset U of IR^2 and let M be the graph of f.

For a point $p = (x_0, y_0, f(x_0, y_0))$ of M, exhibit a vector basis of Tp(M) and give the matrix of Lp with respect to the vector basis of Tp(M) of your choice.

7. For the ellipsoide $\frac{x^2}{3} + \frac{y^2}{4} + z^2 = 1$, (i) obtain the principal directions and principal curvatures at the point $(\sqrt{3}, 0, 0)$ (ii) the Gaussian curvature at (0, 2, 0).

8. (a) (i) Define the notion of a geodesic curve on a smooth surface M and obtain the second order ODE satisfied by the geodesic curves.

(ii) Hence or otherwise, prove that given (1) point p of M, (2), a vector $v \in T_p(M)$ there exists a maximal geodesic $\gamma_{p,v}$ with $\gamma_{p,v}(0) = p$ and $\frac{d}{dt}\gamma_{p,v}(t) = v$.

(iii) Deduce $\gamma_{p,sv}(t) = \gamma_{p,v}(st)$ whenever both the sides of the equation are defined.

(b) Obtain all the geodesic curves on a sphere.

M.Sc. (Mathematics) Part – II
Graph Theory
Paper - IV
April - 2015

QP Code : 18855

Scheme A (External)]	(3 Hours)	**[Total Marks:100**
Scheme B (Internal)]	(2 Hours)	**[Total Marks: 40**

Instructions:

- **Scheme A** students should attempt **any five** questions.
- **Scheme B** students should attempt **any three** questions.
- **All** questions carry **equal marks**.
- **Mention** clearly the **Scheme** under which you are appearing.

1. (a) Prove that graph is bipartite if and only if it has no odd cycle.
   (b) Prove necessary part of Erdos-Gallai conditions.

2. (a) Prove that the center of a tree consists of a single vertex or two vertices joined by an edge. Illustrate the proof.
   (b) State and prove Kruskal's algorithm for finding a minimum weight spanning tree.

3. (a) Prove that the matching in a graph G is maximum if and only if G contains no M augmenting path.
   (b) How many perfect matchings tree can have ? Justify.

4. (a) Explain edge connectivity, vertex connectivity and minimum degree with one example each.
   (b) State Menger's theorem and give one of its application.

5. (a) Prove that graph is Eulerian if and only if it is connected and has no odd vertex.
   (b) Prove that a connected graph is isomorphic to its line graph if and only if it is a cycle.

6. (a) Show that there is no graph with chromatic polynomial $\lambda^3 - 4\lambda^2 + 3\lambda$.
   (b) Prove that the edge chromatic number of a bipartite graph is equal to its maximum degree.

7. (a) Prove that the graphs K_5 and $K_{3,3}$ are non-planar.
   (b) State and prove the Eulers formula for a planar graph.

8. (a) Define Ramsey Number $R(p, q)$ for $p, q \geq 2$.
   Show that $R(p, q) \leq R(p-1, q) + R(p, q-1)$ if $p, q \geq 3$.
   (b) Prove that flow f in a network N is a maximum flow if and only if N contains no f incrementing path.

WE-Con. 8407-15.

M.Sc. (Mathematics) Part – II
Numerical Analysis
Paper - IV
April - 2015

QP Code : 18723

External (Scheme A) (3 Hours) [Total Marks:100
Internal (Scheme B) (2 Hours) [Total Marks:40

Note:

(1) External (Scheme A) students answer any five questions.

(2) Internal (Scheme B) students answer any three questions.

(3) All questions carry equal marks. Scientific calculator can be used.

(4) Write on top of your answer book the scheme under which you are appearing.

Que. 1 (a) Define: Absolute error, Relative error and Percentage error. Round off the number 102.34267 to two decimal places and find the percentage error.

(b) Convert the decimal fraction $(237.4375)_{10}$ to the binary form and then convert to the hexadecimal form.

Que. 2 (a) Derive the Chebyshev iteration formula to find a root of the algebraic or transcendental equation $f(x) = 0$.

(b) Perform two iterations of the Birge-Vieta method to find a root (correct upto three decimal places) of the equation $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$. Use initial approximation $p_0 = 0.5$.

Que. 3 (a) Describe the Jacobi's method to obtain eigen values and eigen vectors of a real symmetric matrix $A = [a_{ij}]$ of order $n \times n$.

(b) Use Crout's method to solve the following system of linear equations.

$x + y + z = 1$
$3x + y - 3z = 5$
$x - 2y - 5z = 10.$

Que. 4 (a) Estimate the error in Newton's forward difference interpolation formula.

(b) Find the maximum or minimum value of y from the following data.

x :	0	1	2	3
y :	0	3.3	10.6	21.9

Que. 5 (a) Derive Newton-Cotes quadrature formula and use it to derive Simpson's three eighth rule for numerical integration.

(b) Evaluate $\int_1^2 \int_1^2 \frac{dx\, dy}{x^2 + y^2}$ using Trapezoidal rule with $h = k = 0.5$.

Que. 6 (a) Obtain the least squares approximation of second degree for $f(x) = \frac{1}{1 + x^2}$ on $[-1, 1]$ with respect to the weight function $w(x) = 1$.

(b) Explain the term Discrete Fourier Transform (D.F.T.) and compute the (4-point) D.F.T. of the sequence $x = (1, 2, 3, 4)$.

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WE-Con. 1609-15.

Que. 7 (a) Derive the Adams-Bashforth corrector formula to solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

(b) Using Runge-Kutta method of fourth order, find an approximate value of $y(1.1)$ and $y(1.2)$, given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ with $y(1) = 1$.

Que. 8 (a) Derive a numerical method (Crank-Nicolson's method) to obtain the numerical solution of one dimensional heat equation.

(b) Write the one dimensional wave equation in the form of difference equation and classify the partial differential equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} = 0$.

M.Sc. (Mathematics) Part – II
Functional Analysis
Paper - V
April - 2015

QP Code : 50780

Scheme A (External)	(3 Hours)	Total Marks: 100
Scheme B (Internal)	(2 Hours)	Total Marks : 40

Instructions:
- Scheme A students should attempt **any five** questions.
- Scheme B students should attempt **any three** questions.
- All questions carry equal marks.
- **Mention** clearly the **Scheme** under which you are appearing.

1. (a) Show that every finite dimensional subspace Y of a normed linear space X is complete.
   (b) Show that a compact subset of a normed linear space is closed and bounded. Is the converse true? Justify with example.
2. (a) Let Y be a closed subspace of a normed space X. For $x + Y \in X/y$, define $|||x+Y||| = \inf\{||x+y|| \mid y \in Y\}$. Show that $|||.|||$ is a norm on X/y.
   (b) Show that any linear functional on $\mathbb{R}^n$ is continuous.
3. (a) State and prove Hahn Banach Theorem.
   (b) Let X, Y be Banach spaces and suppose $T : X \to Y$ is an injective bounded linear map. Prove that image(T) is closed in Y if and only if $\exists$ a constant $M > 0$ such that $||T(x)|| \geq M||x||$, for all $x \in X$.
4. (a) Let $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers. Let $p > 1$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then prove that $\sum a_i b_i \leq (\sum a_i^p)^{1/p} (\sum b_i^q)^{1/q}$.
   (b) A subspace Y of a Banach space X is complete if and only if set Y is closed in X.
5. (a) Let B be a Banach space and let M, N be closed linear subspaces of B such that $B = M \oplus N$. Let $z = x + y$ be the unique representation of a vector in B, where $x \in M, y \in N$. Prove that the mapping $P(z) = x$ is a projection on B whose range and null spaces are M and N.
   (b) If x and y are any two vectors in an inner product space then show that $| < x, y > | \leq ||x||.||y||$.
6. State and prove Closed Graph Theorem.
7. (a) Let M be a closed vector subspace of a Hilbert space H. Show that $(M^{\perp})^{\perp} = M$.
   (b) Show that a maximal orthonormal basis is a complete orthonormal basis for a Hilbert space H.
8. (a) State and prove Uniform Boundedness Theorem.
   (b) Let T be a bounded linear operator on Hilbert space H. Show that its adjoint T^* is also bounded and linear.
