

UNIVERSITY OF BOMBAY
DEPARTMENT OF ECONOMICS

**CHAOTIC ECONOMIC DYNAMICS AND THE BUSINESS
CYCLE**

BY
M.J.MANO HAR RAO

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M.J. Manohar Rao
University of Bombay

1. Introduction

Ever since Koopmans (1947) severely criticized the book Measuring Business Cycles by Burns and Mitchell (1946) as being "measurement without theory" and advocated the development of the structural system-of-equations as an alternative to the study of such dynamics, the study of business cycles ceased to be an active area of economic research.

However, the paper Understanding Business Cycles by Lucas (1977) was instrumental in bringing business cycles back into the mainstream of economics and, as a result, seeking the answer to the fundamental question as to why, in market economies, output and prices undergo repeated fluctuations around trend has become one of the most outstanding challenges to economic research.

Among the several types of models used by macroeconomists to study such dynamics has been vector autoregressions (VARs) which had once become extremely popular. However, the use of VARs to provide evidence on the theories of business cycles has become controversial because of the widely held belief that empirical results from estimated VARs are neither robust nor stable.

Views on the robustness of VAR results owe their genesis to the seminal work of Sims (1980) who used monthly data to estimate a VAR with four variables, i.e., output, the price level, money and interest rates, in order to obtain empirical evidence on how these key variables evolve and influence each other over time.

Sceptics, however, have questioned the robustness of these 'findings and thus VAR evidence in general (see Rankle 1987, Spencer 1989). They have argued that VAR statistics are very sensitive to even minor modifications in the VAR's random and non-random components and therefore any evidence provided by a VAR model on the dynamic relationships amongst its variables is suspect, although the findings of Todd (1990) prove the contrary.

The aim of this paper is to examine the possibility of evolving an alternative approach to analyze business cycles within the framework of macroeconomic theory which we shall use for the formulation of a model based on nonlinear dynamics. Such nonlinear systems sometimes display themselves as attractors and repellers with other well-defined motions such as periodic*. The difficult cases are the chaotics which in certain interesting instances behave as chaotic attractors, where the motion follows a recognizable pattern - reminiscent of business cycles - though always changing within the pattern.

In linear systems there are transients which dissipate, leaving the steady state: for nonlinear chaotic dynamical systems such a dissipation may be non-existent and alternative initial conditions may generate distinctly different long-runs. Under the circumstances, given a chaotic dynamical system which is also subject to exogenous disturbances, if one can correctly discover the true dynamical model, then one can extract the proper deterministic-chaotic component from the signal, leaving the residual as the irregularity due to the shocks. This fact makes it urgent for macroeconomists to consider a certain class of chaotic dynamical models for studying business cycles.

2. Alternative Views of Business Cycles

In 1932, at a Conference on Cycles, representatives from several sciences generally agreed that the following definition (Mitchell 1927; p. 377), "In general scientific use, the word (cycle) denotes a recurrence of different phases of plus and minus departures, which are often susceptible to exact measurement", as being reasonable accurate for all the sciences.

We now know how to construct model economies whose equilibria display business cycles such as those envisioned by Mitchell. For example, a line of research that gained attention in the 1980s demonstrated that cyclical patterns of this form result as equilibrium behaviour for economic environments with appropriate preferences and technologies (see Benhabib and Nishimura 1985). Burns and Mitchell would have been more influential if business cycle theory had evolved along these line- a point made by Koopmans (1957, pp. 215-216) in his largely unnoticed "second thoughts" on their earlier work.

It is now clear that the theory of business cycles has moved in an altogether different direction from the one hoped by Mitchell. Theories with stochastic cyclical laws of motion and seem to have considerable potential for accounting for business cycles; but they have failed to do so. For this reason the theory of nonlinear dynamical systems - which has been hailed as the third great scientific revolution of the 20th century, along with relativity and quantum mechanics (see Devaney 1989) - in which the behaviour of an undisturbed chaotic system closely mimics that of a linear, stochastic system subject to exogenous shocks may be much more appropriate for a study of business cycles.

2.1 The Role of Exogenous Shocks

As early as the 1930s, economists had already developed business cycle models that incorporated difference equations with random shocks (see Frisch 1933; Slutsky 1937). Frisch, however, was careful to distinguish between impulses in the form of random shocks and their subsequent propagation over time. In contrast with proponents of modern business cycle theory, he emphasized damped oscillatory behaviour with the concept of equilibrium being interpreted as a system at rest.

The role of random shocks was emphasized by Adelman and Adelman (1959) who showed that the Klein-Goldberger model, with random shocks, generated aggregate time series that looked remarkably like those of the post-World War II economy of the United States. On the other hand, the deterministic version of this model converged almost monotonically to a point. This exercise forcefully, albeit unfortunately, demonstrated that a stochastic process can generate recurrent cycles which its deterministic counterpart was incapable of doing.

In a scathing indictment of such a procedure, Goodwin (1990, p. 10), remarked, "Frisch misled a generation of investigators by resolving the problem with exogenous shocks, whereas already in the 1920s van der Pol had shown (as Frisch should < have known) that a particular form of nonlinear theory was the appropriate solution". With this blunt challenge, equating probabilistic analysis to a confession of partial ignorance, Goodwin has thrown down the gauntlet to orthodox macroeconometric modeling and has provoked an explosion of theorizing along heterodox lines which hopefully should increase our knowledge about business cycles.

2.2 Modern Business Cycle Theory

In the 1980s and now in the early 1990a, business cycles have increasingly become a focus of study in aggregate economics. Such studies are generally guided by perceived business cycle regularities. But if these perceptions are not, in fact, the regularities, then these lines of research are misguided.

For example, the myth that the price level is procyclical largely exploded by the work of Kydland and Prascott (1990) accounted for the prevalence of studies in the 1970s that used equilibrium models with monetary disturbances or price surprises as the only plausible source of fluctuations in the economy.

Currently, the emphasis in business cycle theory seems to have shifted from pure theoretical work to quantitative theoretical analysis. This trend has been strongly motivated by the empirical regularities noticed in business cycles and the need to provide a theoretical rationale in order to explain them. Along with this shift in focus, aggregate analysis has also undergone a methodological revolution. Previously, empirical knowledge had been organized in the form of equations whose parameters were estimated on the basis of a given data set. It was Sargent (1981) who broke away from this tradition and led the development of tools for inferring values of parameters characterizing our empirical knowledge of preferences, technology, information structure and policy rules, given the behaviour of aggregate time series. Consequently, measurements and quantitative findings in other fields can be used to restrict models of business cycles and render our understanding about the quantitative importance of cyclical disturbances more precise.

A Fresh Look at Traditional Cycle Models

The Great Depression of the 1930s prompted the emergence of precise, quantitative business-cycle analysis. The two best known examples were the Hansen-Samuelson multiplier-accelerator model and the related Lundberg-Meltzer inventory cycle model. Both these were, however, linear and, hence, required shocks to explain the persistence and irregularity of business cycles.

While later alternative solutions in the form of nonlinear dynamics were provided by Kaldor, Hicks and Goodwin, these procedures yielded stable persistent limit cycles, and were unable to explain the presence of irregularities. The revival of the concern about business cycles in the 1970s brought forth a fresh generation of cycle models which being, however, linear, needed exogenous shocks to explain the existence of cycles. Unfortunately, parallel developments in econometric methodology by adopting a linear dynamic structure within the standard system-of-equations approach did not break away from this tradition leaving all irregularity to be attributed to shocks.

It was the belated discovery by Smale (1967) of the seminal work carried out by Lorenz (1963) on the concept of strange attractors (see Ruelle and Takens 1971) as an explanation of the unpredictability of the weather that has forced economists to consider a number of novel approaches to the analyses of cycles. These have been able to achieve two fundamental results of prime importance to economics: firstly, that motions can become bounded so that they are globally stable implying, secondly, that the behaviour is a periodic with a degree of irregularity depending on the initial conditions as well as the parameters.

3.1 Structural and Dynamical Stability: The Rossler Band

Of late, investigations into chaos have attempted to unravel the basic and essential structure by invoking the structural form known as the "Rossler Band" (see Goodwin 1990) which has been applied to a wide variety of problem. The Rossler Band is a three-variable differential equation model which is given by:

$$\dot{x} = -dy - ez \quad (3.1)$$

$$\dot{y} = +hx + fy \quad (3.2)$$

$$\dot{z} = +b + gz(x - c) \quad (3.3)$$

The coefficients of x and y in eqs. (3.2) and (3.1) being $+h$ and $-d$, respectively, the solutions are necessarily oscillatory and are in deviations from the fixed-point equilibrium which represents a steady-state growth. Setting $\dot{z} = 0$, we obtain the results: $\dot{x} = -dy$ and $\dot{y} = hx + fy$, so that:

$$\dot{x} - fx + dhx = 0 \quad (3.4)$$

If $f < 0$, the result is a stable cycle around steady growth. With all parameters positive, an unstable cycle results. In this manner, one can create the type of problem which the Rossler Band is designed to solve chaotically. Such a formulation of chaos theory is, in a certain sense, an elaboration of the discovery of the limit cycle by Poincare. The limit cycle is fundamental to dynamics since without it there is no explanation of the origin or existence of cycles. The Poincare concept was that, given an unstable system around equilibrium, there must arise a level of one or more of the variables, at which a non-linearity occurs, converting a local instability into a global stability. Thus, there is necessarily one or more closed curves (the limit cycles) which separate the region of stability from that of instability.

The Rossler model by means of a simple variation on this ; theme has yielded some very important results. By defining the variable z as a control parameter and dynamically coupling it to the other two state variables x and y in the manner specified in eqs. (3.1) - (3.3), the model ensures that when the variable x expands beyond c , it does not induce a nonlinearity producing a bifurcation from instability to stability, but rather sets into motion the growth of the variable control parameter which progressively inhibits the expansion of the system variables.

Thus, it does not generate a system bifurcation at a particular state level, but rather one in the control -parameter, which has the consequence of creating a band in state space with an outer and inner band. Within that so-called Rossler Band, the system is free to vibrate exhibiting a wide range of motions from a slightly irregular to a wildly chaotic fashion: it can vary in amplitudes, in upper and lower turning-points, as well as in periodicities - all in a seemingly random fashion, although, in fact, it is completely deterministic. Such a kind of chaotic attractor owes its conceptualization to the original Poincare conjecture by being dynamically stable in the sense that, if it is initially outside the Rossler Band, it will eventually end up inside, and if initially within the band, it will remain within.

This important feature provides some encouragement to business cycle theorists as well as econometricians grappling with the problem of analyzing economic time series because it underscores the need to discover the structure of the model which produces the behaviour, so that it is possible to extract the chaotic element and thereby isolate the exogenous stochasticity.

3.2 Chaotic and Aperiodic Behaviour: The van der Pol Model

The investigations into chaotic and aperiodic behaviour owe their genesis to Lord Rayleigh and van der Pol who developed upon Poincare's original consideration of fixed saddle-points in the formulation of limit cycles. In his analysis of sound, Lord Rayleigh considered how a single musical note (and hence a limit cycle) is produced by forcing a vibrational instrument. Then, in the 1920s, the development of vacuum-tube oscillators led to the van der Pol model - where stability was controlled by, the rate of change rather than by the level as in the earlier model - which is basically a two-variable differential equation model given by:

$$\dot{x} = y \tag{3.5}$$

$$\dot{y} = -a(x^2 - 1)y - bx \tag{3.6}$$

Yielding the van der Pol equation

$$\ddot{x} + a(x^2 - 1)\dot{x} + bx = 0 \tag{3,7}$$

Thus, the van der Pol model will obviously produce a limit cycle, being unstable for any $|x| < 1$ and stable for any $|x| > 1$. The quadratic nonlinearity, however, produces special behaviour: for small values of a , the cycle is approximately sinusoidal; but for larger and larger values, the result approaches a square wave which is appropriately termed as a relaxation oscillator.

By imposing a suitable exogenous cycle on the van der Pol model, one can produce erratic, chaotic or aperiodic behaviour which can be as irregular as desired. It is, however, to be distinguished from chaotic attractors because its behaviour produces a reasonably clear and sequential pattern. This quasi periodic motion will never, barring special cases, produce exact repetition and hence it becomes infinitely dense on the torus,

4. A Nonlinear Model of Business Cycles

Based on empirically tented macroeconomic phenomena, we propose to theoretically integrate a van der Pol forcing function onto a Rossler model basically involving just two variables, i.e., growth rates and inflation rates, in order to investigate the following basic aspects of their cyclical behaviour: (1) Their amplitude of fluctuations and (2) The degree of procyclical and counter cyclical movement of prices with real output.

Model selection, or the underlying rationale behind its specification, was guided essentially by the fact that although the classic work on hyperinflation by Cagan (1956) is a unique attempt to study monetary phenomena, it does suggest the need for an alternative approach to try and explain the circumstances under which prices, after a prolonged period of stable inflation, could explode suddenly into a hyperinflationary spiral in the manner of a bifurcation which is well in keeping with reality.

Although this phenomenon has received some attention from economists in the past (see Sargent 1982), most attempts to try and figure out the central causes for such a process of bifurcation have evolved essentially around: (i) the role of inflation tax revenue, over borrowing, debt servicing, commodity prices, political instability, capital flight, amongst others, to explain why the inflation rate can suddenly explode (see Sachs 1987); and (ii) the role of macroeconomic stabilization policies, comprising orthodox programs (of fiscal austerity) as well as heterodox programs (of wage-price controls), to explain why the inflation rate can suddenly be brought under control, albeit at great cost to the economy (see Helpman and Leiderman 1988).

4.1 The Hypothesis

However, these approaches seem to be unsatisfactory because" they concentrate on too many exogenous variables and ignore one of the central propositions of monetary economics: A sustained increase in money growth reduces long-run real money demand.

In order to examine the importance of the above proposition and ascertain the role that money growth and money demand play in determining the existence, uniqueness and stability of the equilibrium rate of inflation, we assume the following money demand function on the lines suggested by Dornbusch (1989):

$$(M/Py) = \alpha - \beta \pi \quad (4.1)$$

where M is nominal money supply, P is the price level, y is real output, π is the rate of inflation; and $\alpha, \beta > 0$.

In terms of time derivatives, eq. (4.1) can be written as:

$$\dot{M} = \alpha(\dot{P}y + P\dot{y}) - \beta(P\dot{y}\pi + P\dot{y}\pi + P\dot{y}\pi) \quad (4.2)$$

by ignoring second-and third-order interactions. Given $\dot{y}/y = g$ (real output growth rate) and $\dot{P}/P = \pi$ (inflation rate), yields:

$$\dot{\pi} = \beta^{-1}[\alpha g + \alpha \pi - \beta g \pi - \beta \pi^2 - (\dot{M}/Py)] \quad (4.3)$$

Rewriting \dot{M}/Py as the product of (\dot{M}/M) and (M/Py) , we have:

$$\dot{\pi} = \beta^{-1}[\alpha g + \alpha \pi - \beta g \pi - \beta \pi^2 - \mu(\alpha - \beta \pi)] \quad (4.4)$$

obtained by setting $\mu = \dot{M}/M$ and by replacing (M/Py) by eq. (4.1).

Now, in steady state, $\dot{\pi} = 0$, implying that eq. (4.4) reduces to a quadratic in π given by the following expression:

$$\beta \pi^2 - (\alpha - \beta g + \beta \mu)\pi - (\alpha g - \alpha \mu) = 0 \quad (4.5)$$

yielding the following steady state solutions for it:

$$\pi_1 = \alpha/\beta \quad \text{and} \quad \pi_2 = \mu - g \quad (4.6)$$

highlighting the role played by money demand as well as money and output growth rates in deriving the equilibrium inflation rate.

Now, stability theory requires that for an equilibrium solution to be stable $\dot{\delta\pi}/\delta\pi < 0$. From eq. (4.4), we have:

$$\dot{\delta\pi}/\delta\pi = (\alpha/\beta) + \mu - \varrho - 2\pi \quad (4.7)$$

which, when evaluated at the two equilibrium solutions, yields:

$$\left. \frac{\dot{\delta\pi}}{\delta\pi} \right|_{\pi=\alpha/\beta} = (\mu - \varrho) - (\alpha/\beta) \quad (4.8a)$$

and

$$\left. \frac{\dot{\delta\pi}}{\delta\pi} \right|_{\pi=\mu-\varrho} = (\alpha/\beta) - (\mu - \varrho) \quad (4.8b)$$

The stability conditions indicate that the solution towards which inflation rates would converge would always be the larger one, i.e., if $\alpha/\beta > \mu - \varrho$, then α/β would be the stable solution and vice versa. This analysis shows that money growth rates, real output growth rates and the parameters of the money demand function have a large bearing on the stability of inflation and, thus, any study on the analytical foundations of business cycle behaviour should largely concentrate on these links.

Qualitatively identical results to the ones just obtained are seen to emerge if we follow the tradition of Sargent and Wallace (1.9A1.) and adopt the money demand function given by:

$$(M/P) = \alpha - \beta\pi \quad (4.9)$$

where M/P is real money demand, π is the rate of inflation and the parameters α and β are assumed to be positive.

In terms of time derivatives, eq. (4.9) can be written as:

$$\dot{M} = \alpha\dot{P} - \beta(\pi\dot{P} + \dot{\pi}P) \quad (4.10)$$

by ignoring second-order interaction terms. This yields:

$$\dot{\pi} = \beta^{-1}[\alpha\pi - \beta\pi^2 - (M/P)] \quad (4.11)$$

which, as before, by rewriting $M/P = (M/MMM/P) = \mu(\alpha - \beta\pi)$ is:

$$\dot{\pi} = \beta^{-1}[\alpha\pi - \beta\pi^2 - \mu(\alpha - \beta\pi)] \quad (4.12)$$

Setting $\eta = 0$ in eq. (4.12) yields the following expression:

$$\beta \pi^2 - (\alpha + \beta \mu) \pi + \alpha \mu = 0 \quad (4.1.3)$$

from which we derive the steady state solutions for π given by:

$$\pi_1 = \alpha/\beta \quad \text{and} \quad \pi_2 = \mu \quad (4.14)$$

Prom eq. (4.11), we have:

$$\dot{\delta \pi} / \delta \pi = (\alpha/\beta) + \mu - 2\pi \quad (4.15)$$

which, when evaluated at the two equilibrium solutions, yields:

$$\left. \frac{\dot{\delta \pi}}{\delta \pi} \right|_{\pi = \alpha/\beta} = \mu - \alpha/\beta \quad (4.16a)$$

and

$$\left. \frac{\dot{\delta \pi}}{\delta \pi} \right|_{\pi = \mu} = \alpha/\beta - \mu \quad (4.16b)$$

Once again, the above results on the nature of stability of these two equilibria indicate that the solution towards which inflation rates would converge would always be the larger one. 4.2

4.2 The Evidence

In order to substantiate our results, consider Table 1 where we have listed some recent highly inflationary experiences.

Table 1
Recent High-Inflation Experiences
(Percent per year)

Year	Argentina		Bolivia		Brazil		Nicaragua		Peru	
	μ	π	μ	π	μ	π	μ	π	μ	π
1981	54	104	15	29	65	106	27	24	47	75
1982	196	165	121	133	82	98	26	25	37	64
1983	288	344	199	269	95	142	67	31	76	111
1984	523	627	791	1281	141	197	90	35	89	110
1985	650	672	7574	11750	275	227	163	220	199	163
1986	144	90	297	276	178	145	252	681	175	78
1987	98	131	62	15	144	230	637	912	96	86
1988	234	343	30	16	238	682	7406	10205	261	667
1989	2751	3080	19	15	748	1287	•••	23710	1586	3399
1990	1629	2314	22	17	•••	2938	•••	•••	••••	7482

Source: International Financial Statistics (1991)

It is seen that, although orthodox macroeconomics insists on a long-run correspondence between the rate of money growth (μ) and the rate of inflation (π), there is little evidence of such a money-inflation link in Table 1, leading us to believe that that inflation rates could be converging, not towards the **conventional** equilibrium solutions given by $\pi = \mu$ or $\pi = \mu - g$ but rather towards the alternative equilibrium solution given by $\pi = \alpha/\beta$

In order to examine this possibility for all the countries listed in Table 1, we need to estimate money demand functions for each of them. As reliable data on real growth rates was not available in many cases, it was decided to fit a money demand function of the type suggested by eq. (4.9), i.e., $M/P = \alpha - \beta\pi$, and contrast, for each country, the two solutions, i.e., $\pi_1 = \alpha/\beta$ and $\pi_2 = \mu$. Given our analysis, the actual inflation rate for each country should be closer to the higher of the two solutions.

To do so, we initially estimate a Cagan-type money demand function for each country of the form: $M/P = Ae^{-b\pi}$ in order to determine the 'optimal'¹ inflation rate (defined as $\pi^* = 1/b$) at which the inflation tax revenue is the maximum. This is necessary because once π^* has been exceeded, the real monetary base falls faster than the rise in the inflation rate leading to a reduction in the inflation tax revenue which, in turn, suddenly increases budget deficits and money creation thereby leading to a 'bifurcation' in the inflation rates. In order to see if our model could predict such bifurcations, it was decided to truncate the sample period before such a bifurcation point.

In Table 2, we provide the parameters of the estimated money demand function: $\ln(M/P) = \ln A - b\pi$ for all the five countries.

Table 2
Optimal Inflation Rates And Truncation Points

Country	In A	b	R^2	π^*	T
Argentina: (1975-90)	7.9450 (80.7)	-0.0003240 (-3.3)	0.40	3086	1988
Bolivia: (1975-90)	8.4372 (79.8)	-0.0000797 (-2.2)	0.21	12542	1984
Brazil: (1975-89)	4.9366 (69.5)	-0.0018056 (-10.0)	0.8f1	554	1987
Nicaragua: (1975-88)	3.5552 (38.1)	-0.0001019 (-3.0)	0.38	9814	1987
Peru: (1975-89)	11 .2390 (132.0)	-0.0004404 (-4.6)	0.59	2271	1988

Notes:

- (1) Real money stock (M/P) was measured as follows:
Argentina: Thousands of Australes at 1980 prices.
Bolivia: Bolivianos at 1975 prices. Brazil:
Thousands of Cruzieros at 1975 prices. Nicaragua:
Thousands of Cordobas at. 1985 prices. Peru:
Thousands of Tntis at 1975 prices.
- (2) π^* , i.e., the optimal inflation rate, is equal to $1/b$.
- (3) T, i.e., the year in which the inflation rate exceeded π^* , is the truncation point of the sample while estimating eq.(4.9).
- (4) The period indicated in brackets under each country refers to the estimation period for each equation.
- (5) Figures in brackets under each coefficient are t-statistics.

Based upon these truncation points, we now estimate the money demand function, eq. (4.9), for each country. The results are provided below. In each case, the years in parentheses refer to the sample (comprising only the period of high-inflation prior to the truncation point), inclusive of lags. The figures in parentheses below each coefficient value indicate t-statistics. Also reported are the R^2 , the standard error of regression (SF.R), the Durbin-Watson statistic (DW) and the log likelihood (T.T.). Also. Included are the coefficients of first-order and second-order serial correlation, AR (1) and AR (2); to which are appended in parentheses the corresponding Breusch-Godfrey test statistic.

Argentina: (1975-88)	M/P = 2157.6611 - 0.9172 π (7.4) (-2.6)		
R* = 0.8190; DW = 1.7367;	SER = 253.19; LL = -81.00;	AR(1) = 0.1856 (0.54) AR(2) = 0.3765 (1.58).	
Bolivia: (1977-84)	M/P = 8580.2209 - 0.7788 π (6.0) (-1.2)		
R* = 0.9588; DW = 2.8186;	SER = 307.72; LL = -39.59;	AR(1) = 0.7919 (1.67); AR(2) = 0.5661 (0.97).	
Brazil : (1975-87)	M/P = 118.8782 - 0.2158 π (2.5) (-2.2)		
R = 0.8840; DW = 1.9675;	SER = 11.70; LL = -40.18;	AR(1) = 0.9919 (2.37); AR(2) « -0.1512 (0.39).	
Nicaragua: (1981-87)	M/P = 46.7938 - 0.0055 π (4.9) (-0.1)		
R' = 0.7717; DW = 3.0966;	SER = 9.77; LL. = -14.47;	AR(1) = 0.6651 (1.26); AR(2) = -1 .2747 (1.00).	
Peru: (1982-88)	M/P = 58876.3720 -20.0911 π (7.9) (-0.5)		
R = 0.8063; DW = 3.2258;	SER = 8875.54; LL = -48.53;	AR(1) = 1.0077 (2.51); AR(2) = -1.6158 (2.31);	

Using these estimated functions, we now forecast the equilibrium inflation rate for all these countries and compare it with the actual one in order to assess our analytical framework.

Table 3

Country	Reference Year	Predicted And Actual Inflation Rate's			Actual Inflation Rate (π)
		Equilibrium Solutions: α/β	μ	Stable Solution Max [$\alpha/\beta, \mu$]	
Argentina:	1989	2352	2751	2751	3080
Bolivia:	1985	11017	7574	11017	11750
Brazil:	1988	551	238	551	682
Nicaragua:	1988	8508	7406	8508	10205
Peru:	1989	2931	1586	2931	3399

Notes:

- (1) The reference year is T+1 where T is the truncation point.
- (2) α/β Is obtained from the corresponding money demand equation.
- (3) The money growth rate (μ) and the actual inflation rate (π) correspond to the reference year.

4.3 The Model

The results indicate that the actual inflation rate in the year of bifurcation for each of the five countries is quite close to the stable solution predicted by our model. Considering that we were carrying out an out-of-sample prediction, the fact that we were able to accurately predict the extent of bifurcation does support our hypothesis that the eventual course of inflation would be determined jointly by money (and output) growth rates as well as the parameters of the money demand function.

Thus, the short-run dynamics of inflation would, be governed by the nonlinear difference equation, eq. (4.4), i.e.,

$$\dot{\pi} = \beta^{-1} [\alpha g + \alpha \pi - \beta g \pi - \beta \pi^2 - \mu(\alpha - \beta \pi)] \quad (4.17)$$

while eventual long-run inflation which is a proxy for expected inflation (π^e) would be given by:

$$\pi^e = \max \{ \alpha/\beta, \mu - g \} \quad (4.18)$$

In order to incorporate variable growth rates, we extend the model by including a Lucas-type output supply function given by:

$$y = y_0 e^{nt} (P/P^e)^\tau \quad (4.19)$$

where P is the actual and P^e is the expected price level. By differentiating eq. (4.19) with respect to time, we obtain:

$$g = \dot{y}/y = n + \tau(\pi - \pi^e) \quad (4.20)$$

By analyzing the resulting 3-equation model comprising eqs. (4.17), (4.18) and (4.20) in inflation (π), expected inflation (π^e) and growth (g), it would be possible to try and provide some answers to the fundamental question raised by Todd (1990, p. 34) as to whether, in general, models that use only generic variables like the money stock, the price level, or output can by themselves provide an adequate foundation for empirical work.

5. The Coupled Rossler-van der Pol Model

The coupled equations of the nonlinear model are as follows:

$$\dot{\pi} = \beta^{-1} [\alpha g + \alpha \pi - \beta g \pi - \beta \pi^2 - \mu(\alpha - \beta \pi)] \quad (5.1)$$

$$\pi^e = \max [\alpha/\beta, \mu - g] \quad (5.2)$$

$$g = n + \tau(\pi - \pi^e) \quad (5.3)$$

where it is seen that eqs. (5.1) and (5.3) are structurally similar to eqs. (3.6) and (3.3) of the van der Pol and Rossler models, respectively. Hence, our system can be considered as a hybrid of both these approaches. Since the Rossler model is already a chaotic attractor, it is not obvious what one can expect by impressing an endogenous van der Pol cycle upon it using the concept of forced oscillators.

However, while the solution of a one-dimensional difference equation with fairly simple nonlinearities could be extremely complex, this is not true for one- or even two-dimensional systems of differential equations. In these cases, chaos cannot arise because of the Poincare-Bendixon theorem (see Hirsch and Smale 1974) which states that any limit of a two-dimensional system of differential equations is either a fixed point or a cycle. This is the reason as to why we discretize the above system and work with its difference-equation analog, i.e., by replacing $\dot{\pi}$ by $\delta\pi (= \pi - \pi_{-1})$, because the differential equation model as specified above cannot per se display chaos.

To understand the nature of the system dynamics, as well as to visualize this kind of coupled limit cycle systems, we could initially consider the limit cycle generated by π and g . In such a two-dimensional phase space, we would have a simple limit cycle attractor, which could also be a fixed-point attractor.

Likewise, we could then consider the limit cycle generated by it" and q . In such a two-dimensional phase space, we would once again have a simple limit cycle attractor. When these two limit cycles interact, the dimension of the phase space increases and the previously independent limit cycles now become fastened to each other. With a phase space built of these three variables, the ensuing limit cycle is more complex as q is affected by both π and π^e . Thus, its limit cycle would oscillate in two frequencies and the result of the first limit cycle being swept around by the second in three-dimensional phase space would generate a torus attractor.

The coupled motion of these three interacting variables has two degrees of freedom and, following the Poincare-KAM conjecture (see Arnold 1963, Kolmogorov 1979), the surface of the resulting three-dimensional torus will be the attractor for such a system provided that the frequencies of the coupled system are qua H_i - periodic. If, however, such a condition is not satisfied, then with each orbit around the torus the perturbation is amplified and the result is resonance. Mathematically, this amplification causes the surface of the torus to explode in phase space and while these three variables are still attracted to this surface and -try to reach it, the effort of doing so causes their orbits to snap causing them to spin off chaotically. It is possible to hypothesize that the Great Depression in the United States (1)29-33) and the hyperinflations in Germany (1922-23) and Hungary (194.5-46) were the result of these three variables seeking the structure of an attractor, called the strange attractor, whose surface had been fragmented across the phase space.

Studies have proved, that there do exist certain critical values for economic parameters at which the limit cycle of the system jumps from one attractor to another and that a strange attractor could be encountered at any of these critical points of instability, referred to as Hopf instability. The existence of such critical points implies that, in any study of business cycle behaviour, the irregular behaviour of cycles should not be attributed entirely to exogenous shocks because such shocks are arbitrary and can be made to explain any kind of behaviour. Under the circumstances, it becomes imperative to consider other sources of irregularities such as the ones considered in this study, especially now that strange attractors have been discovered, in order to model the dynamics of business cycles.

6. The Chaotic Dynamics of Business Cycles

6.1 Business Cycles

In order to empirically test some of our conjectures, we have to infer numerical estimates for each of the parameters. Setting $\alpha=0.2$, $\beta=1$, $\tau=1.8$, $\mu=0.5$ and $n = 0.05$ which are fairly reasonable estimates, we simulated the model with $\pi(0)=0.5$ as the initial condition. The results of the simulation for the first 100 time periods are provided in Figure 1.

The results clearly mimic a business cycle with well-defined turning points separating the peaks from troughs, which are interposed by periods of stability. What is equally interesting is that while, by and large, prices are counter cyclical, they exhibit a fleeting procyclical nature during certain phases of the business cycle. This phenomenon will be examined in greater detail when we construct the attractors of such systems.

Figure 1
BUSINESS CYCLES

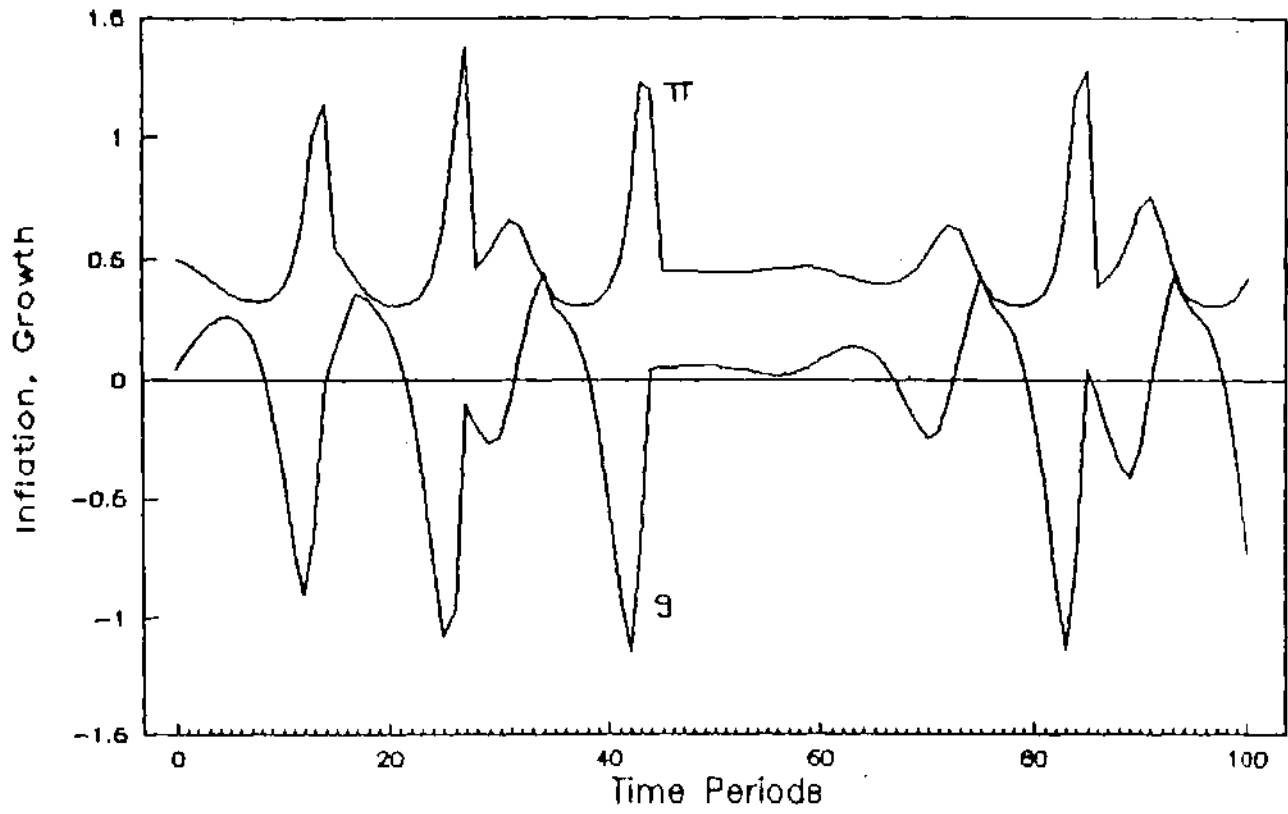
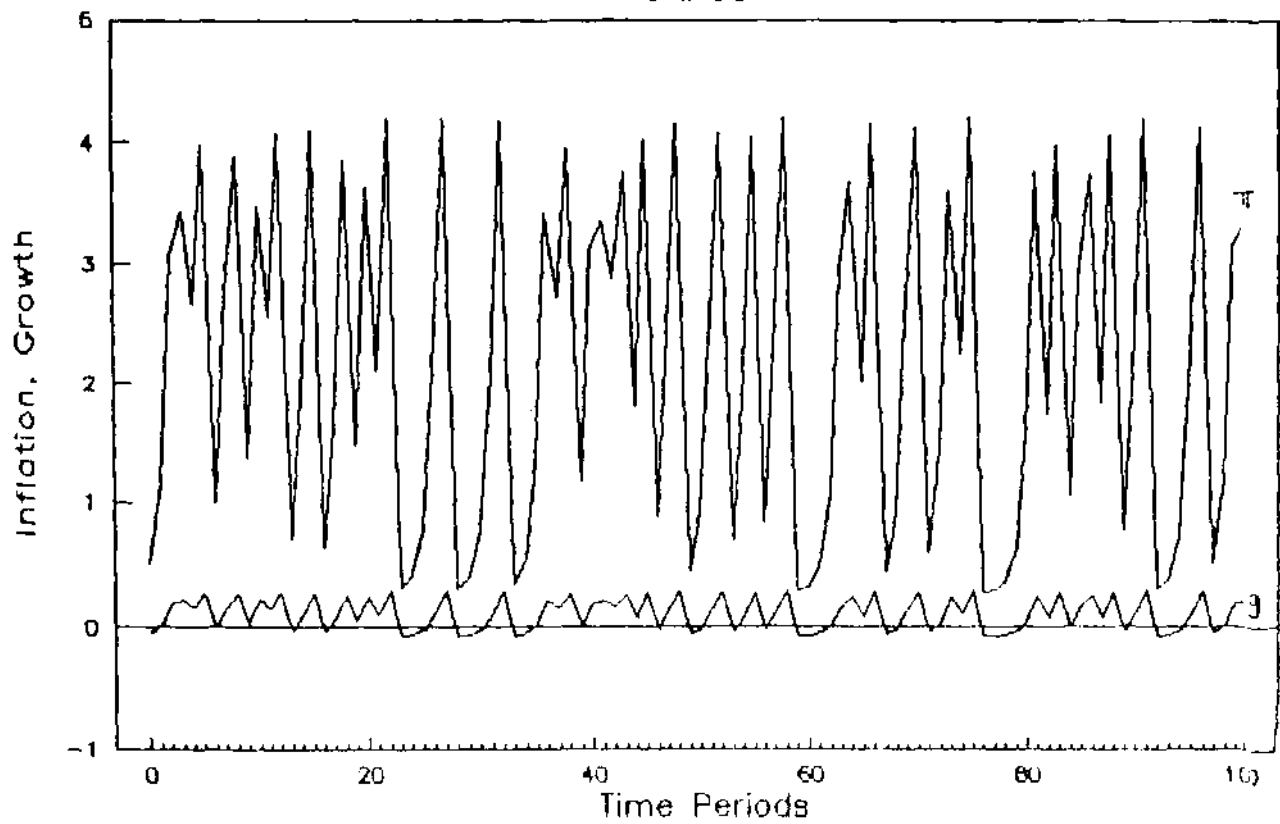


Figure 2
CHAOS



6.2 Period-Doubling Routes To Chaos

The emergence of chaos is evident, for the following set of parameters: $\alpha=3$, $\beta=1$, $\tau=0.1$, $\mu=0.2$ and $n=0.2$. Based upon these estimates, we once again simulated the model with $\pi(0)=0.5$ as the initial condition. The results of the simulation for the first 100 time periods are provided in Figure 2.

The extreme volatility in inflation and growth rates which is a hallmark of chaos is evident here. What is interesting is that the route to chaos is via "period-doubling" bifurcations where π and g oscillate between fixed points, nailed a limit cycle, for given values of a parameter: in our case, for $\beta=1.5$, they oscillate between 2 values (two-cycle); for $\beta=1.24$, between 4 values (four-cycle); for $\beta=1.184$, between 8 values (eight-cycle); and at $\beta=1$, between n values (n -cycle) which is chaos.

6.3 Temporary Equilibrium

One of the characteristics of chaos is that minute changes in a parameter can bring about substantive changes in the time paths of the variables. In order to test this proposition, we repeat the last experiment with identical values for all the parameters, except money growth which is set at a fractionally higher level, i.e., $\mu=0.21$. The results of the simulation for the first 100 time periods are provided in Figure 3.

While the trajectories fluctuate as chaotically as before, there is a brief period of absolute inability in π and g very similar to the "windows" appearing in the Feigenbaum attractor) with the latter clinging on to the natural rate of growth of output (i.e., $n = 0.2$) a la Lucas. However, this is temporary episode as the trajectories become chaotic soon thereafter.

Figure 3
TEMPORARY EQUILIBRIUM

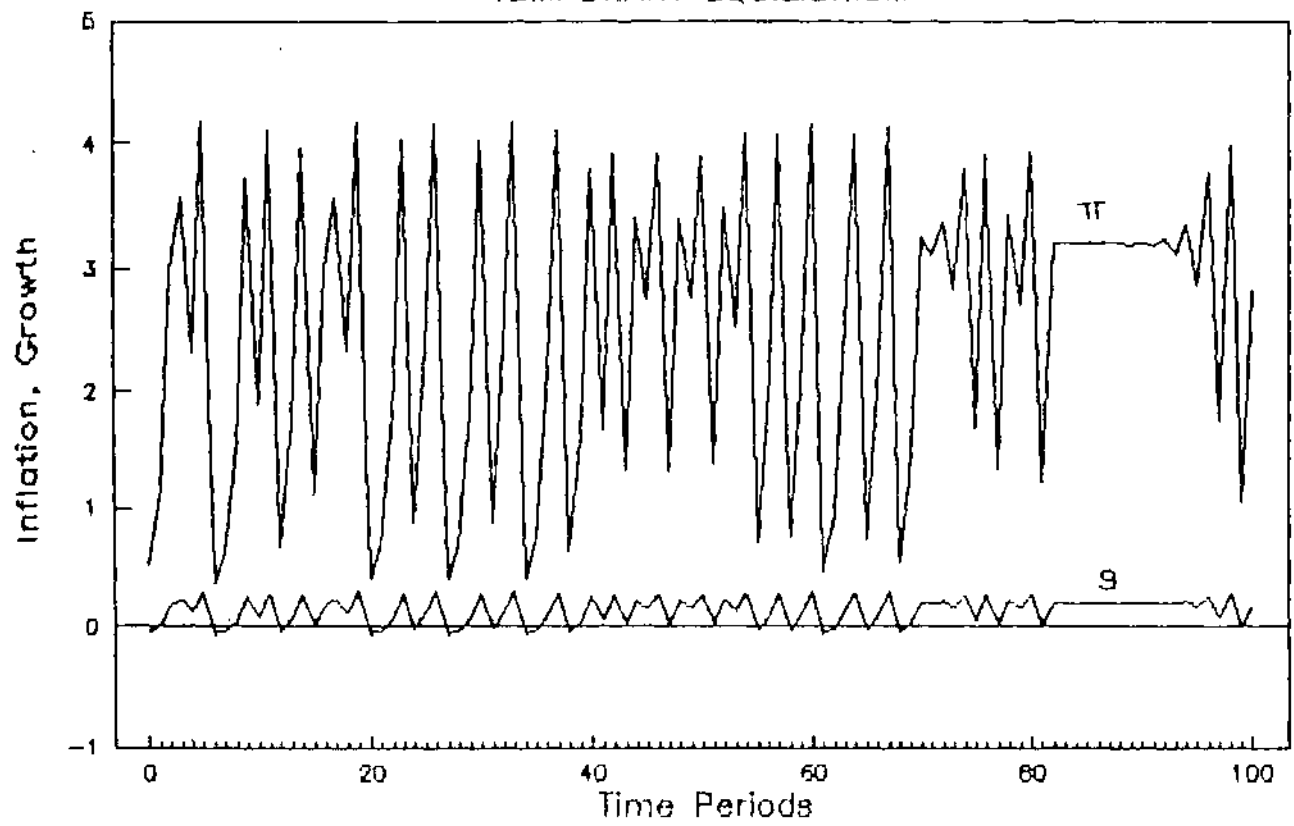
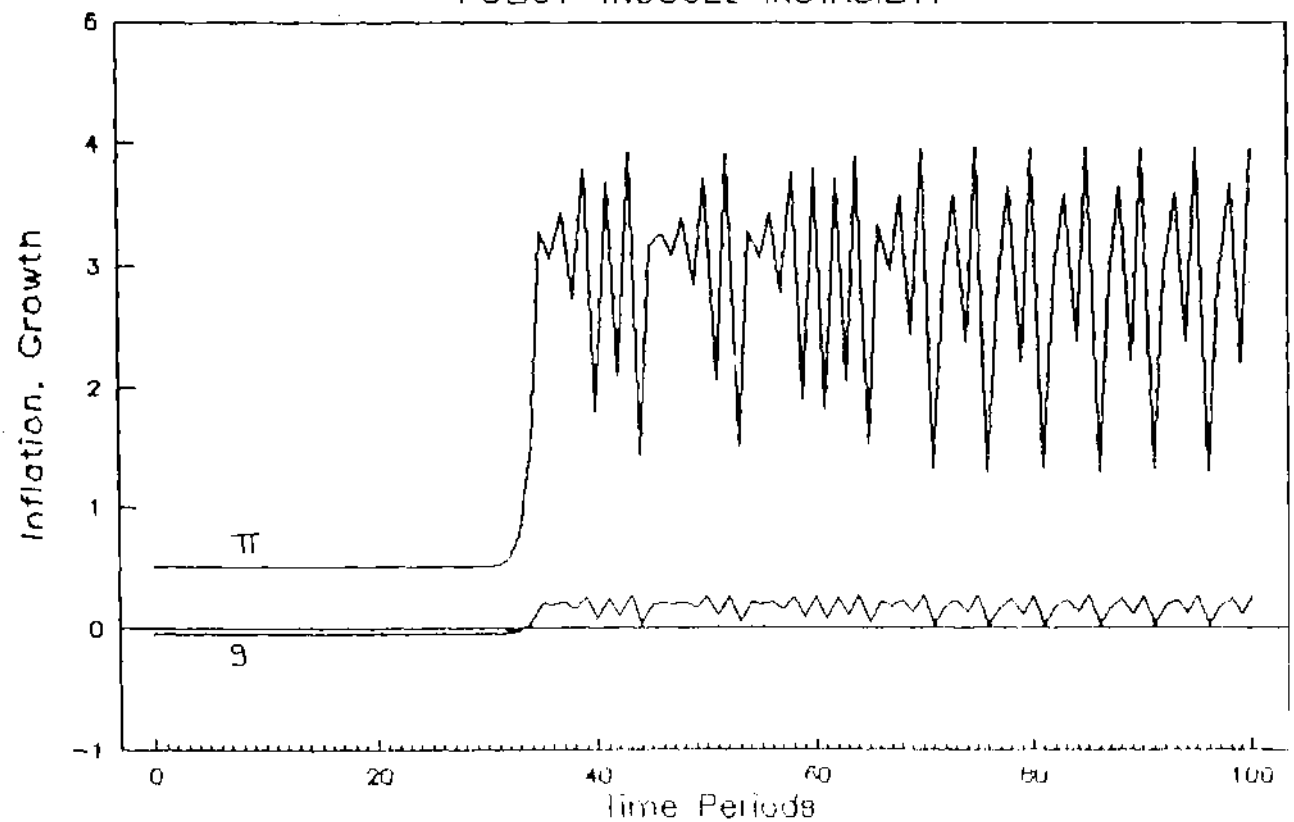


Figure 4
POLICY-INDUCED INSTABILITY



6.4 Policy-Induced Instability

One of the "paradoxes" of macroeconomics is that, under certain assumptions, a higher growth rate of money can actually reduce inflation. This directly follows from a Cagan-type of money demand function: $(M/P) = e^{-\beta\pi}$. logarithmic differentiation of this expression with respect to time, yields: $\dot{\pi} = \beta^{-1}(\pi - \mu)$, from which we obtain: $\dot{\delta\pi}/\delta\mu = -\beta^{-1}$ which proves the point.

Using the same set of parameters as in the last experiment, but setting $\mu=0.45$ indicates (see Figure 4) that while both π and g are low initially, they flare up chaotically later on. More importantly, once chaos sets in, the minimum inflation rate is 1*26 (i.e., 126%) as against the corresponding minimum of 28% in Experiment 1, suggesting that monetary expansion would yield higher inflation in the long-run after suppressing it initially.

6.5 Reconstructed Attractors

Figure 5 shows the $n-g$ projections of the solution of the system whose parameters were specified in Section 6.1. It was noticed that the resulting attractors are quite similar across a broad range of initial conditions, $\pi(0)$. Thus, although the time paths of chaotic systems exhibit sensitive dependence on initial conditions, their attractors remain qualitatively unchanged.

This is true of all dissipative systems. Broadly speaking, dissipation implies the loss of memory of initial conditions, so that they play no role in the dynamics of the system once this has reached its asymptotic regime. Unlike conservative systems, dissipative dynamical systems are characterized by contraction of phase space volumes with increasing time. This property will be shown to be true for our system as well.

Figure 5
RECONSTRUCTED ATTRACTOR

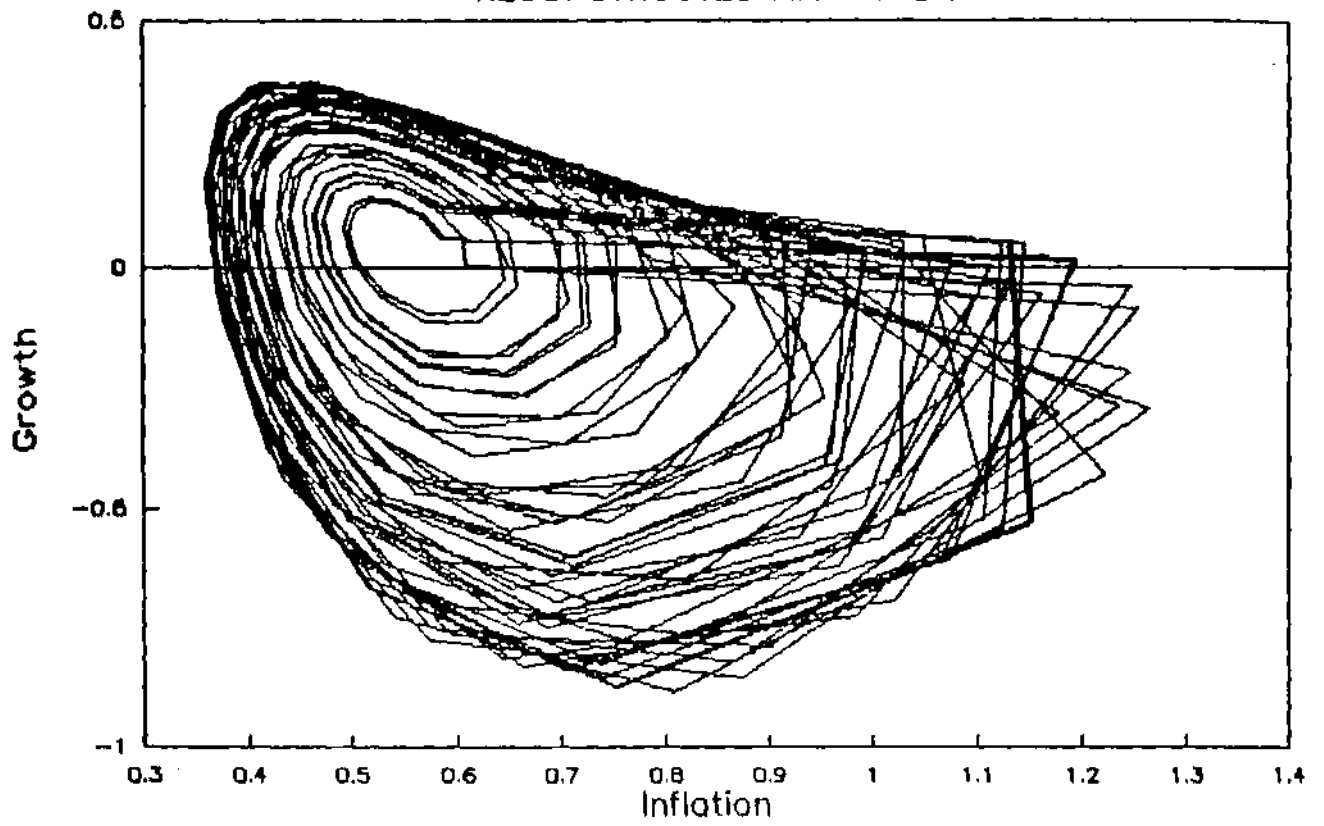
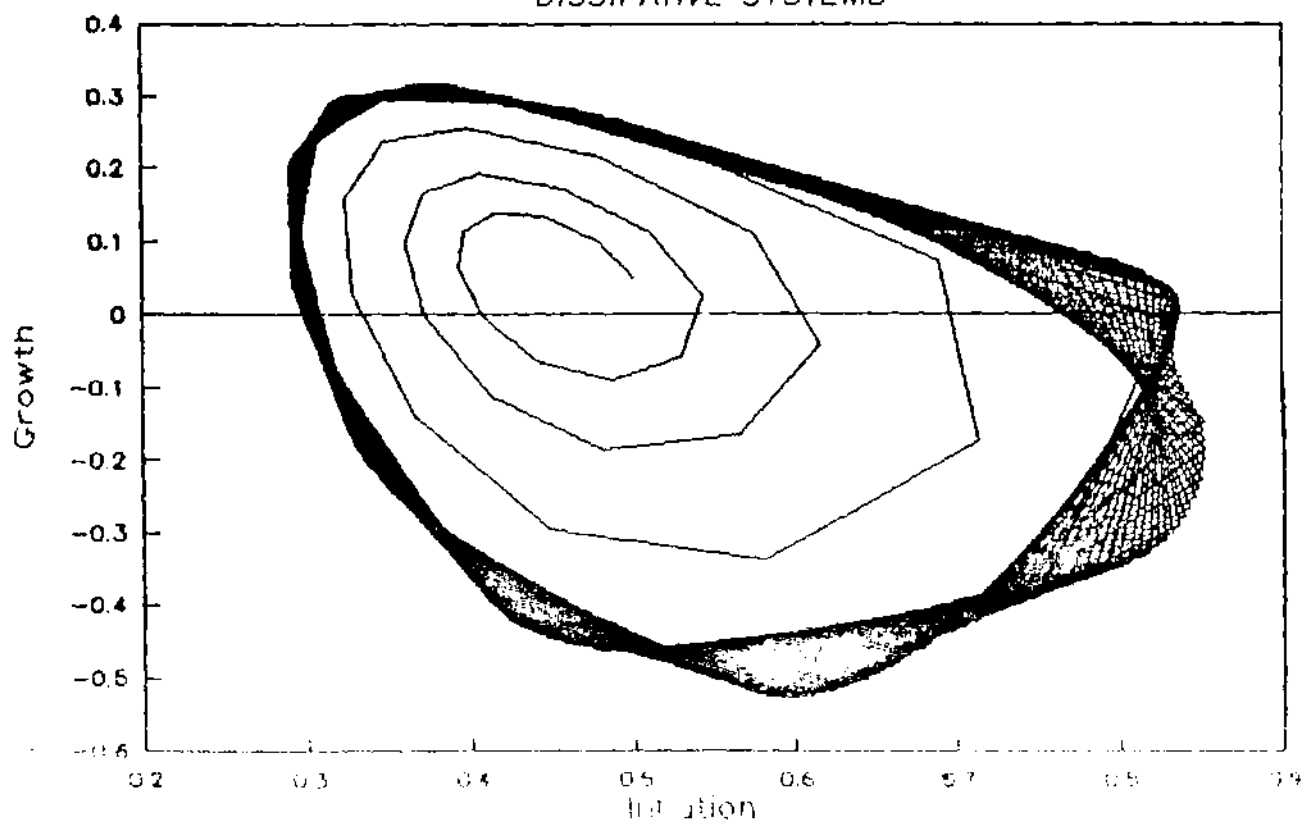


Figure 6
DISSSIPATIVE SYSTEMS



6.6 Dissipative Systems

Setting $\alpha=0.2$, $\beta=3$, $\tau=1.8$, $\mu=0.5$, $n=0.05$ and $\pi(0)=0.5$; and **simulating** the .system, we obtained its π - q projection (Figure 6). It is clearly evident that the system is dissipative because its trajectories converge towards a limit cycle attractor. Formally, a system is dissipative if all its Lie derivatives are negative. In our case, this would imply that $\delta\pi/\delta\pi = [\alpha + \beta(\mu - q) - 2\beta\pi] < 0$; or $\pi > [(\alpha/\beta) + (\mu - q)]/2$. Considering that π eventually converges to the larger of either α/β or $\mu - q$, it follows that the above condition is satisfied and that our system is indeed dissipative. In conservative systems, such as the LVG (Lotka-Volterra-Goodwin) model of cyclical economic growth, there are no attracting limit cycles, and initial conditions determine which of the infinite number of cycles is actually followed.

6.7 Closed Orbits and the Spiral Attractor

As mentioned earlier, the route to chaos is via period-doubling bifurcations which are usually the result of changes in critical parameters. In our case, β , the sensitivity of money demand to inflation, and n , the natural rate of growth of output, are two such parameters. Retting $\beta=12$ and $n=0.04$, all other parameters unchanged vis-à-vis the last experiment, yields a two-period cycle whose attractor is a closed orbit (Figure 7); while setting $\beta=21$ yields the so-called "spiral attractor" (Figure 8), which is substantially different from most of the other attractors in the literature, e.g. the Lorenz attractor, the Henon attractor, the Feigenbaum attractor, in as much as the dissipative nature of the system, i.e., the contraction of phase space volumes with increasing time, is clearly brought out.

Figure 7
CLOSED ORBITS

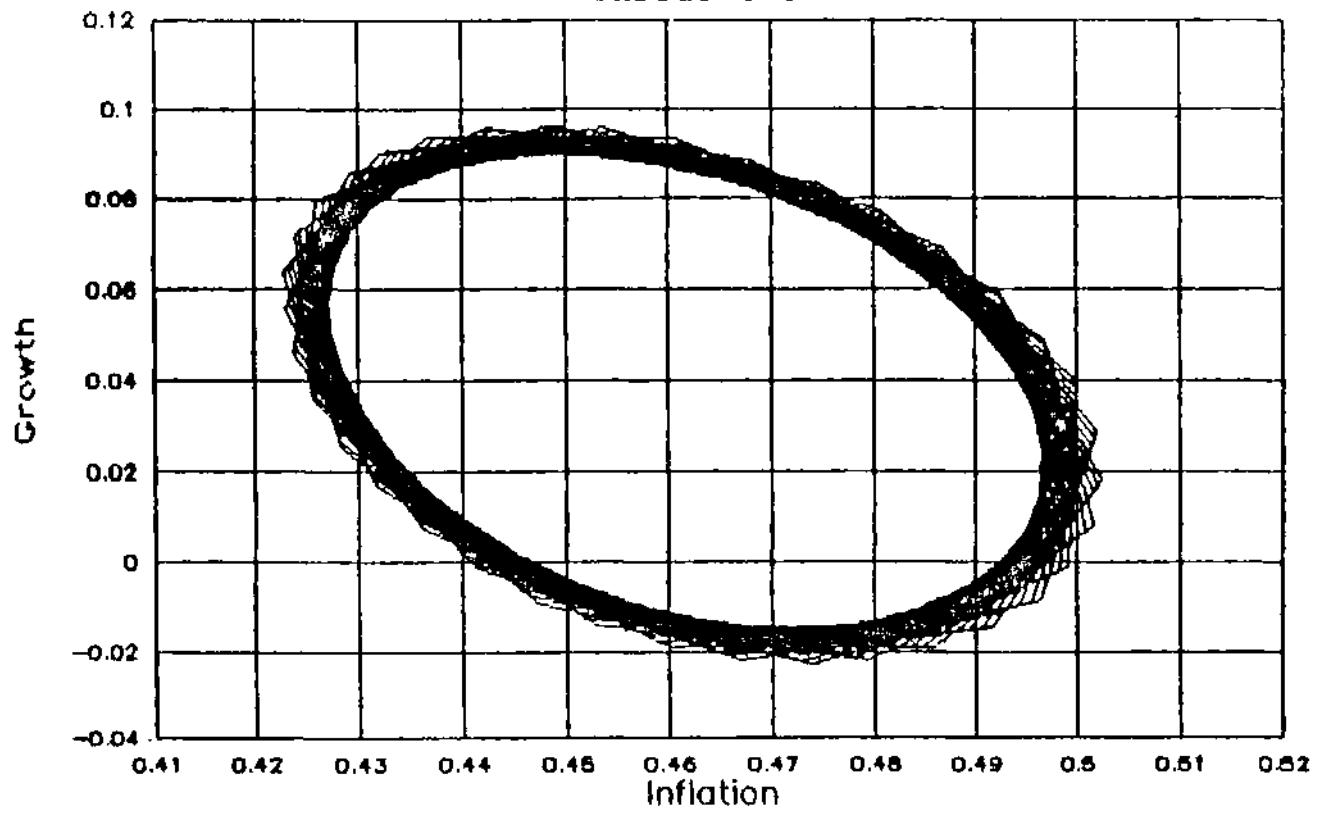
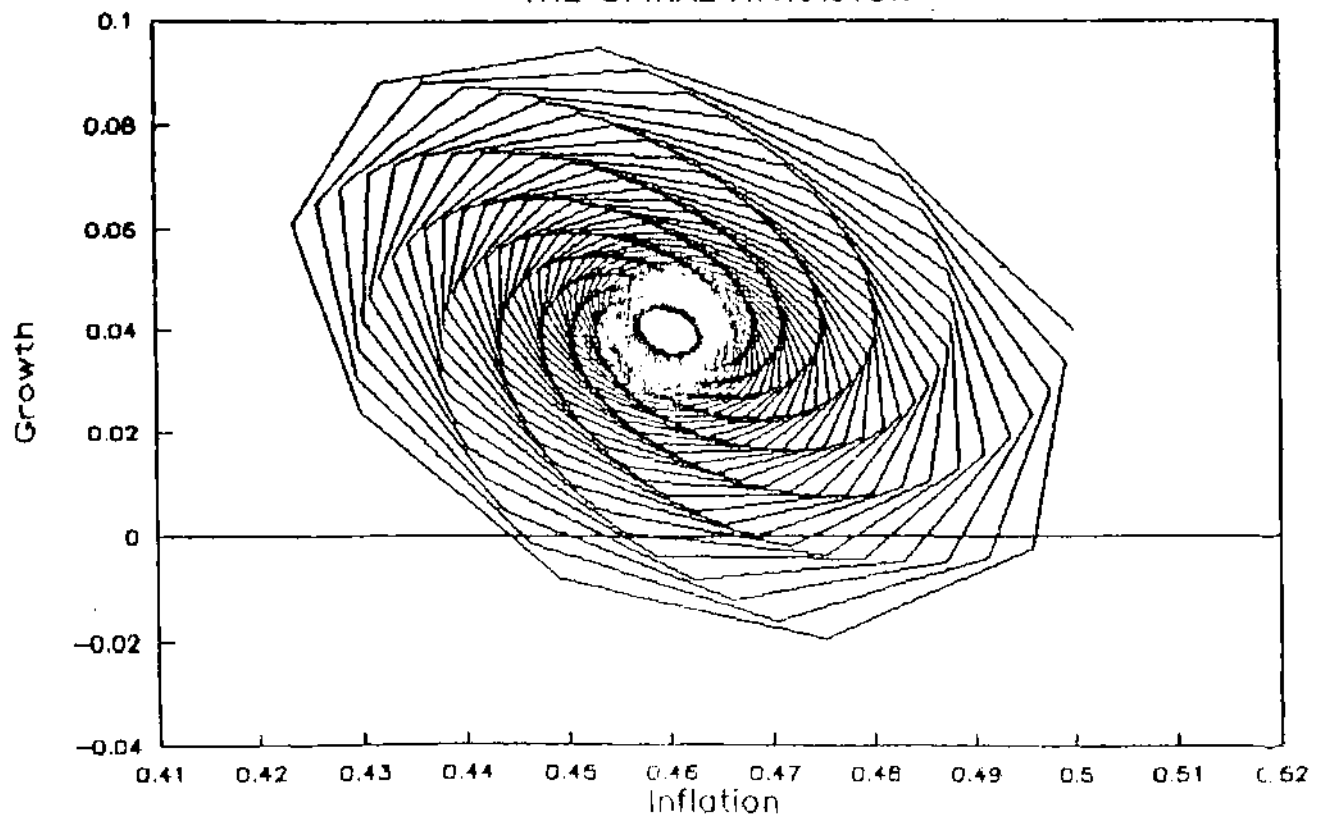


Figure 8
THE SPIRAL ATTRACTOR



What is interesting in both cases is that: (i) prices are seen to be both pro- as well as counter cyclical along different phases of the business cycle and (ii) output growth rates never converge to their natural growth rates ($n=0.04$) as predicted by Lucas but cycle around it indefinitely. Such behaviour is increasingly evident whenever period-doubling bifurcations occur. If we define θ_{n-1} , θ_n , θ_{n+1} as successive period doubling points, then the ratio: $\delta = \{(\theta_n - \theta_{n-1}) / (\theta_{n+1} - \theta_n)\}$ is termed as the Feigenbaum number which is equal to 4.6692. What is interesting to note is that based on the period doubling points noticed in Section 6.2, we have: $\theta_{n-1}=1.5$; $\theta_n=1.24$; and $\theta_{n+1}=1.184$. These values when substituted into the above expression yields $\delta=4.6429$ which, being very close to the predicted Feigenbaum number, clearly highlights the chaotic nature of our dynamical system.

7. Structural Stability and Dynamic Instability

Within the framework of our model, as $\delta\pi/\delta g = \alpha - \beta\pi$, while $\delta g/\delta\pi = \tau > 0$; the solutions of the system could be either oscillatory (if $\pi > \alpha/\beta$) or non-oscillatory (if $\pi < \alpha/\beta$). This coupled to the fact that expected inflation is determined in a manner analogous to a "flip" bifurcation, while output supply is positively affected by a rise in unanticipated inflation ($\pi - \pi^e$), makes the model structurally stable but dynamically unstable, thereby answering Kolmogorov fundamental criticism of Volterra's structurally unstable formulation.

The (van der Pol) forcing function which is dynamically coupled to the (Rossler) band links up g with π and such a coupling obviously induces a limit cycle which is stable for any:

$$g > \{\beta\pi^2 - \alpha\pi - \beta\mu\pi + \mu\alpha\} / \{\alpha - \beta\pi\} \quad (5.4)$$

and unstable if otherwise. The form of this quadratic nonlinearity imposed by this function could, however, produce a special type of behaviour where the limit cycle bear no resemblance to each other if β exceeds a critical value.

The characteristics and degree of irregularity depend on the ratio of exogenous and endogenous periodicity, as well as on the degree of nonlinearity in the model, as determined by its parameters. If β is small, then the model is rather sinusoidal and the impressed wave form will merely alter the sinusoid as demonstrated by Goodwin (1990). However, rising values of β would significantly alter the shape of the wave and increase its variability, emphasizing the relaxation aspect of the endogenous oscillation. If the parameter μ is reduced, implying an attenuated forcing function, the considerable alterations in the phase portrait would increase the variability of the wave shape.

Given such dynamically unstable cycles, the usual solution, following Poincare, is to assume upper and lower nonlinearity, yielding global stability and at least one closed limit cycle - a single equilibrium motion from a fixed point equilibrium which could represent a steady-state growth path. While Poincare generalized the concept of equilibrium from a point to a closed curve, Lorenz extended this result further by generalizing the closed curve to a closed region (see Smnle 19R0). All models from Rayleigh onwards have, by introducing two nonlinearity, posited such upper and lower values to convert local instability into global stability. Goodwin (1951) realized that one non-linearity was sufficient to provide a limit cycle, thereby demonstrating that two nonlinearities were not a necessary condition.

What. is more interesting in that Rossler found that he needed only one non-linearity to stabilize the vastly more erratic dynamics of chaotic motion. We realize, in the context of our model, that the Rossler Band - a chaotic attractor most comprehensible and applicable to economics - instead of defining an upper and lower bound for the two key variables under consideration, i.e., the inflation rate and the real growth rate, posits a "control" variable, i.e., expected inflation, which provides a growing upward (downward) pressure on output whenever the actual inflation rate is higher (lower) than the two possible expected inflation rates specified in advance. This effectively stabilizes the system globally rendering it free to move locally within the Rossler Band defined around the zero equilibrium.

Coupling our Rossler model with a suitably imposed exogenous cycle induced by the van der Pol forcing function, it is possible to generate erratic, chaotic or aperiodic behaviour, which can be as irregular as desired. It is, however, to be distinguished from strange attractors because, as the study of Rao and Bhogle (1990) showed, as long as points of Hopf instability area not encountered during the simulation, the quasi-periodic motion generated by the system will never, barring special cases, produce exact repetition and hence becomes infinitely dense on the torus. Given such a structure, it would be theoretically possible to predict the behaviour of our system using the notion of asymptotic predictability - meaning that even if we are ignorant about the exact position of our system currently, we are confident that, no matter how far into the future we predict, it will be moving somewhere on the surface of our three-dimensional torus.

8. Conclusion

This feature could be a considerable source of encouragement to macroeconomists who often need to know the extent of error associated with their long-range forecasts. It would also then be possible to examine the validity of the following four elements of a distinctive version of monetarism called monism (see Poole 1978): (1) Monetary policy, or its instability, is the primary cause of business cycles, (2) The time path of money supply is a good indicator of monetary policy, (3) Changes in money supply are the primary cause of business cycles because these changes cause, lead, and are positively related to changes in output (at least in the short run), and (4) Changes in money supply lead, are positively related-to, and are the primary determinants of changes in the price level (at least in the long run).

Nonlinear dynamic systems theory has been successfully employed in several fields of research, notably fluid dynamics. Many of the phenomena described earlier, e.g. period-doubling sequences, chaotic motion, have been observed in fluid motion. However, fluid dynamics has one great advantage over economics in as much as the basic equations of motion, i.e., the Navier-Stokes equations, as well as the basic equations for turbulence, i.e., the Prandtl-Kolmogorov equations, are known. While it is true that these are capable of generating extremely complicated behaviour which, in many cases are unsolvable, the fact remains that in economics, we have to try and explain observed phenomena, which can yield trajectories as complicated as fluid flow, without recourse to any known economic equations of motion. This is the major drawback facing economic prediction currently.

Economies also contains the following paradox. In microeconomics, all economic variables are Been to be generated by the rational decisions of maximizing agents, thereby implying that all microeconomic variables can be assumed to be completely deterministic. However, in macroeconomics, most of these aggregated variables are frequently viewed as being random. The problem is to reconcile the fact that the same variables are deterministic and random at the same time - a paradox which may be explained if the economic system is chaotic.

Currently, a transition towards a chaotic regime has been witnessed in several macroeconomics around the world. Many industrialized countries seem to have suddenly switched from conditions of steady-state growth (which was predominantly the domain of traditional economic theory) towards those exhibiting erratic fluctuations. As a result, it has become very difficult to predict several key' macroeconomic variables. While it is still not certain that what we are observing is chaotic dynamics because there are too many independent sources of noise which affect economic data, it is possible to conjecture that some of the observed noise is not extrinsic noise independent of the economic system hut is intrinsic noise generated by the chaotic dynamics of the system itself.

It is because of such a reconciliation between microeconomics and macroeconomics; between extrinsic noise and intrinsic noise; and, ultimately, between chaos and order that leads one to believe that nonlinear dynamic systems theory can provide a far better description of economic behaviour than that which is currently possible using alternative paradigms.

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