Ph.D - Ente. test. - science

mathematics.

Con. 3209–12.

KK–2594

(3 Hours)

[Total Marks : 100

28/4/12

Section I

Answer all the 20 objective questions. [40]

1. Let $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be non-zero linear maps such that $\ker(f) \subseteq \ker(g)$. Then
   (A) $f = g = 0$
   (B) $\ker(g) = \ker(f)$
   (C) $\frac{\ker(g)}{\ker(f)}$ is isomorphic to $\mathbb{R}$
   (D) none of the above holds.

2. The dimension of the vector space consisting of all linear transformations from $\mathbb{R}^5$ to $\mathbb{R}$ is
   (A) 1 (B) 0 (C) 5 (D) 6

3. The quotient ring $\frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$ is
   (A) a finite field
   (B) not an integral domain
   (C) an infinite field
   (D) an integral domain but not a field.

4. Find a true statement below:
   (A) $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}(\sqrt{3})$ as fields
   (B) $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}(\sqrt{3})$ as vector spaces over $\mathbb{Q}$
   (C) $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 2$
   (D) $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{6})$.

5. Let $f : \mathbb{Z}_6 \longrightarrow \mathbb{Z}$ be a non-zero group homomorphism. Then
   (A) $f = 0$
   (B) f is surjective
   (C) f is injective
   (D) none of the above holds.

6. Let $A = \{\frac{1}{n}\sin(\frac{1}{n}) \mid n \in \mathbb{N}\}$. The find a true statement from below:
   (A) A has one limit point and it is 1
   (B) A has one limit point and it is -1
   (C) A has one limit point and it is 0
   (D) A has three limit points and they are $0, 1, -1$.

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7. Let $A = \{1, 2\} \subset \mathbb{R}$. Let $f(x) = \inf\{|x - a| \mid a \in A\}$ $(x \in \mathbb{R})$. Then
   (A) f is not a continuous function
   (B) f is differentiable on $\mathbb{R} \setminus \{1, 2\}$
   (C) f is differentiable on $\mathbb{R} \setminus \{3/2\}$
   (D) f is differentiable on $\mathbb{R} \setminus \{1, 3/2, 2\}$.

8. Let $\mathcal{F} = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid |f(x) - f(y)| \leq C|x - y| \; \forall \, x, y \in \mathbb{R}$ for some constant $C\}$.
   Find a true statement from below:
   (A) If $f \in \mathcal{F}$, then f is uniformly continuous on $\mathbb{R}$
   (B) If $f \in \mathcal{F}$, then f is differentiable on $\mathbb{R}$
   (C) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable, then $f \in \mathcal{F}$
   (D) none of th above holds.

9. Find a true statement from below:
   (A) $n\log(1 + \frac{1}{n^2}) \longrightarrow 1$ as $n \longrightarrow \infty$
   (B) $n\log(1 + \frac{1}{n+1}) \longrightarrow 1$ as $n \longrightarrow \infty$
   (C) $(n+1)\log(1 + \frac{1}{n}) \longrightarrow 0$ as $n \longrightarrow \infty$
   (D) $n^2\log(1 + \frac{1}{n^2}) \longrightarrow 0$ as $n \longrightarrow \infty$.

10. Find the norm below with respect to which the space $C([0, 1])$ of continuous real valued functions on $[0, 1]$ is complete:
    (A) $||f||_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$
    (B) $||f||_2 = (\int_0^1 f(x)^2 \, dx)^{1/2}$
    (C) $||f||_1 = \int_0^1 |f(x)| \, dx$
    (D) none of the above.

11. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ be a continuous function. In which case below imply that $f = 0$?
    (A) $\int_0^\pi x^n f(x) dx = 0 \; \forall \, n \in \mathbb{N} \cup \{0\}$
    (B) $0 = \int_0^\pi \cos(nx) f(x) \, dx \; \forall \, n \in \mathbb{N} \cup \{0\}$
    (C) $\int_0^\pi \sin(nx) f(x) \, dx = 0 \; \forall \, n \in \mathbb{N} \cup \{0\}$
    (D) $0 = \int_0^\pi (\cos(nx) + \sin(nx)) f(x) dx \; \forall \, n \in \mathbb{N} \cup \{0\}$.

12. Find a true statement below:
    (A) $f(z) = \cos z$ $(z \in \mathbb{C})$ is bounded function.
    (B) There is an non-constant entire function with $f(\mathbb{C}) = \mathbb{R}$.
    (C) There exists a bounded non-constant entire function.
    (D) If f, g are entire functions such that $f(iy) = g(iy) \; \forall \; 0 < y < 1$, then $f = g$.

13. Which function below is uniformly continuous?
    (A) $f(x) = 1/x$ $(0 < x < 1)$
    (B) $f(x) = x^3$ $(x \in \mathbb{R})$
    (C) $f(x) = \sin^2(x)$ $(x \in \mathbb{R})$
    (D) $f(x) = \sin(1/x)$ $(0 < x < 1)$

14. Let $A = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$. Find a set below which is homeomorphic to A.
(A) $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$
(B) $\{(x,y) \in \mathbb{R}^2 \mid x + y = 0\}$
(C) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
(D) $\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$

15. Let
$$f(z) = \frac{e^z + 1}{e^z - 1}$$
$\forall\, z \in \mathbb{C}$. Then $z = 0$ is
(A) a removable singularity of f
(B) a pole of f of order 2
(C) an essential singularity of f
(D) a pole of f of order 1.

16. Let $\gamma : [0,\pi] \longrightarrow \mathbb{C}$ be the contour defined by $\gamma(s) = 2e^{is}\ \forall\, s \in [0,\pi]$. Then $\int_\gamma \frac{z+1}{z}\, dz =$
(A) $i\pi$ (B) $3 + i\pi$ (C) $-4 + i\pi$ (D) 2π

17. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $f : D \longrightarrow \mathbb{C}$ be an analytic function such that $f(0) = 0$. Define $g(z) = \frac{f(z)}{z}\ \forall\, z \in D \setminus \{0\}$ and $g(0) = f'(0)$. Then find the correct statement from below:
(A) g is not continuous at $z = 0$
(B) $z = 0$ is a removable singularity of g
(C) g has a pole at $z = 0$
(D) g is a meromorphic function.

18. Find a compact set below:
(A) $\{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$
(B) $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{1 \le i \le n} x_i^2 = 1\}$
(C) $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{1 \le i \le n} x_i = 1\}$
(D) $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \cdots x_n = 1\}$.

19. Find a true statement below:
(A) If A is a dense subset of a topological space X, then $X \setminus A$ is nowhere dense in X.
(B) If A is no-where dense subset of a topological space X, then $X \setminus A$ is dense in X.
(C) $\mathbb{R} \times \{0\}$ is a dense subset of $\mathbb{R} \times \mathbb{R}$
(D) $\{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is a dense subset of $\mathbb{R} \times \mathbb{R}$.

20. The point in the plane '$x - y - z = 0$' in $\mathbb{R}^3$ which is nearest to the point $(4,-1,1)$ is
(A) $(2,1,1)$ (B) $(1,0,1)$ (C) $(0,4,-4)$ (D) $(5,2,3)$.

Section II

Answer any three questions. [30]

1. Prove that any non-constant analytic map $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an open map.
2. Prove that any compact subset of a Hausdorff topological space X is a closed subset of X.
3. Give an example of a sequence of real valued continuous functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ satisfying the following properties:
   (a) $\int_0^1 f_n(x)\,dx = 1 \;\forall\, n \in \mathbb{N}$ and
   (b) $f_n(x) \longrightarrow 0$ as $n \longrightarrow \infty \;\forall\, x \in [0,1]$.
4. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map such that the minimum polynomial $f(X)$ of T is product of k many distinct linear factors for some $1 \leq k \leq n$. Then prove that T is a daigonalisable linear map.
5. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function. Fix $p \in \mathbb{R}^n$. Prove that there exist constants $r > 0$ and $M > 0$ such that
$$|f(x) - f(y)| \leq M\|x - y\|$$
for all $x, y \in \mathbb{R}^n$ satisfying $\|x - p\| < r,\ \|y - p\| < r$.

Section III

Answer any two questions. [30]

1. Let G be a finite group. For $x \in G$, define its conjugacy class in G. Derive the conjugacy class equation for G. Deduce that a group of order p^2 is an abelian group where $p \in \mathbb{N}$ is any prime.
2. Let $f : \mathbb{R} \longrightarrow [0, \infty)$ be a Lebesgue measurable function. Prove that there exists a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ satisfying the following properties:
   (a) $0 \leq s_n(x) \leq s_{n+1}(x) \;\forall\, n \in \mathbb{N}, \forall\, x \in \mathbb{R}$ and
   (b) $s_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$ for each $x \in \mathbb{R}$.
3. Show that a complex polynomial of degree $n \in \mathbb{N}$ has exactly n many zeroes in $\mathbb{C}$.
4. Prove that ant isometry of $\mathbb{R}^n$ is composite of at most $n + 1$ many reflections.

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